## Math 2 Homework #3

1. The point of this exercise is to give an example of two random variables that have covariance 0, but are not independent. Let $X$ be a random variable uniformly distributed on $[-1, 1]$ and $Y$ be the random variable defined by $Y = X^2$.

   (a) Compute the joint cumulative distribution for $X, Y$. That is, $P\{X \leq x, Y \leq y\}$ as a function of $x, y$.

   (b) Do $X, Y$ have a joint density? If so, what is it? If not, why not? (Hint: in general $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$. Will this work here?)

   (c) Compute the covariance of $X, Y$. (Hint: use $Cov[XY] = E[XY] - E[X]E[Y]$. Notice that $XY = X \cdot X^2 = X^3 = g(X)$ with $g(x) = x^3$. How do you get the expectation of a function of a RV?)

   (d) Compute the probabilities $P\{X > 1/2, Y > 1/3\}, P\{X > 1/2\}, P\{Y > 1/3\}$. Explain why this shows that $X, Y$ are not independent.

2. The point of this exercise is for you to learn how to compute the density of a function of a RV. I will put some such question on the final exam. Recall that in the case of a function of one random variable we have:

   - If $Y = g(X)$ and $g$ has an inverse function $h$, then

   $$f_Y(y) = f_X(h(y))|h'(y)|.$$

   In the case of a function of two random variables we have:

   - If $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$, and we can find inverse functions $h_1, h_2$ so that $X_1 = h_1(Y_1, Y_2), X_2 = h_2(Y_1, Y_2)$, then

   $$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) \det \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix}$$

   (a) Suppose $X = N[0, 1]$ is Gaussian, and $Y = X^3$. Compute the density of $Y$.

   (b) Suppose $X$ has density

   $$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

   calculate the density of $Y$ defined by $Y = 3X - 2$.

   (c) Suppose $X_1, X_2$ have joint density

   $$f_{X_1X_2}(x_1, x_2) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2 - x_2} & \text{if } x_2 > 0 \\ 0 & \text{else} \end{cases}$$

   and $Y_1, Y_2$ are defined by $Y_1 = X_1 + X_2$, and $Y_2 = X_1 - X_2$. Calculate the joint density of $Y_1, Y_2$.

3. Suppose that on average 5.3 cars are observed driving a certain road each minute.

   (a) Explain why it is reasonable to model the number of cars observed on this road in a given minute as a Poisson random variable.

   (b) By definition, if $X$ is a Poisson($\lambda$) random variable, then

   $$P\{X = k\} = e^{-\lambda} \lambda^k / k!$$

   Compute the probability that the number cars driving this road in a given minute is at least two.

4. Suppose $X, Y$ are RV's with joint density

$$f_{X,Y} = \begin{cases} 2e^{-x-y} & \text{if } 0 < x < y \\ 0 & \text{else} \end{cases}$$

   (a) Compute the density of $Y$.
   (b) Compute the conditional density $f_{X|Y}(x|y)$
   (c) Calculate the conditional probability $P\{X > 5|Y = 6\}$
   (d) Compute the conditional probability $P\{X > 5|Y > 6\}$.

5. Suppose that 5% of gamblers in Las Vegas cheat at a particular card game. Assume that cheaters have probability 0.60 of winning each hand, but honest players have only probability 0.45. You observe the player next to you for 20 hands, and he wins 15 of them.

   (a) What is the probability that a cheating player will win 15 out of 20 hands?
   (b) Given your observation of his success, what is the probability that he is cheating?